

## ON THE REAL POLYNOMIAL BOHNENBLUST–HILLE INEQUALITY

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ABSTRACT. It was recently proved by Bayart et al. that the complex polynomial Bohnenblust–Hille inequality is subexponential. We show that, for real scalars, this does no longer hold. Moreover, we show that, if  $D_{\mathbb{R},m}$  stands for the real Bohnenblust–Hille constant for  $m$ -homogeneous polynomials, then  $\limsup_m D_{\mathbb{R},m}^{1/m} = 2$ .

## 1. INTRODUCTION

If  $E$  is a Banach space, real or complex, we say that  $P$  is a homogeneous polynomial on  $E$  of degree  $m \in \mathbb{N}$  if there exists an  $m$ -linear form  $L$  on  $E^m$  such that  $P(x) = L(x, \dots, x)$  for all  $x \in E$ . It is customary to denote by  $\hat{L}$  the restriction of  $L$  to the diagonal of  $E^m$ . An old and widely known algebraic result establishes that for every homogeneous polynomial  $P$  of degree  $m$  on  $E$  there exists a unique symmetric  $m$ -linear form  $L$  on  $E^m$ , called the polar of  $P$ , such that  $P = \hat{L}$ . We denote by  $\mathcal{P}(^m E)$ ,  $\mathcal{L}(^m E)$  and  $\mathcal{L}^s(^m E)$  the spaces of continuous  $m$ -homogeneous polynomials, continuous  $m$ -linear forms and continuous symmetric  $m$ -linear forms on  $E$  respectively. It is well known that homogeneous polynomials or  $m$ -multilinear forms are continuous on  $E$  if and only if they are bounded, respectively, over the unit ball  $B_E$  of  $E$  or  $B_E^m$ . In that case

$$\begin{aligned}\|P\| &:= \sup\{|P(x)| : x \in B_E\}, \\ \|L\| &:= \sup\{|L(x_1, \dots, x_m)| : x_1, \dots, x_m \in B_E\},\end{aligned}$$

define a norm in  $\mathcal{P}(^m E)$  and  $\mathcal{L}(^m E)$  respectively. If  $P \in \mathcal{P}(^m E)$ , we shall refer to  $\|P\|$  as the polynomial norm of  $P$  in  $E$ . This norm is very difficult to compute in most cases, for which reason it would be interesting to obtain reasonably good estimates on it. The  $\ell_p$  norm of the coefficients of a given polynomial on  $\mathbb{K}^n$  ( $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ ) has also been widely used in mathematics and is much easier to handle. Observe that an  $m$ -homogeneous polynomial in  $\mathbb{K}^n$  can be written as

$$P(x) = \sum_{|\alpha|=m} a_\alpha x^\alpha,$$

where  $x = (x_1, \dots, x_n) \in \mathbb{K}^n$ ,  $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$ ,  $|\alpha| = \alpha_1 + \dots + \alpha_n$  and  $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ . Thus we define the  $\ell_p$  norm of  $P$ , with  $p \geq 1$ , as If  $E$  has finite dimension  $n$ , then the polynomial norm  $\|\cdot\|$  and the  $\ell_p$  norm  $|\cdot|_p$  ( $p \geq 1$ ) are equivalent, and therefore there exist constants  $k(m, n), K(m, n) > 0$  such that

$$(1.1) \quad k(m, n)|P|_p \leq \|P\| \leq K(m, n)|P|_p,$$

for all  $P \in \mathcal{P}(^m E)$ . The latter inequalities may provide a good estimate on  $\|P\|$  as long as we know the exact value of the best possible constants  $k(m, n)$  and  $K(m, n)$  appearing in (1.1).

The problem presented above is an extension of the the well known polynomial Bohnenblust–Hille inequality (polynomial BH inequality for short). It was proved in [5] that there exists a constant  $D_m \geq 1$  such that for every  $P \in \mathcal{P}(^m \ell_\infty^n)$  we have

$$(1.2) \quad |P|_{\frac{2m}{m+1}} \leq D_m \|P\|.$$

Observe that (1.2) coincides with the first inequality in (1.1) for  $p = \frac{2m}{m+1}$  except for the fact that  $D_m$  in (1.2) can be chosen in such a way that it is independent from the dimension  $n$ . Actually Bohnenblust

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and Hille showed that  $\frac{2m}{m+1}$  is optimal in (1.2) in the sense that for  $p < \frac{2m}{m+1}$ , any constant  $D$  fitting in the inequality

$$|P|_p \leq D\|P\|,$$

for all  $P \in \mathcal{P}({}^m\ell_\infty^n)$  depends necessarily on  $n$ .

The polynomial and multilinear Bohnenblust–Hille inequalities were overlooked for a long period (see [3]) and were only rediscovered in the last few years; now these inequalities can be seen as an extension of the successful theory of absolutely summing operators (see [7]) and have fundamental importance in different fields of Mathematics and Physics, such as Operator Theory, Fourier and Harmonic Analysis, Complex Analysis, Analytic Number Theory and Quantum Information Theory (see [2–4, 6, 8, 9, 11–14] and references therein).

The best constants in (1.2) may depend on whether we consider the real or the complex version of  $\ell_\infty^n$ , which motivates the following definition

$$D_{\mathbb{K},m} := \inf \left\{ D > 0 : |P|_{\frac{2m}{m+1}} \leq D\|P\|, \text{ for all } n \in \mathbb{N} \text{ and } P \in \mathcal{P}({}^m\ell_\infty^n) \right\}.$$

If we restrict attention to  $\mathcal{P}({}^m\ell_\infty^n)$  for some  $n \in \mathbb{N}$ , then we define

$$D_{\mathbb{K},m}(n) := \inf \left\{ D > 0 : |P|_{\frac{2m}{m+1}} \leq D\|P\| \text{ for all } P \in \mathcal{P}({}^m\ell_\infty^n) \right\}.$$

Note that  $D_{\mathbb{K},m}(n) \leq D_{\mathbb{K},m}$  for all  $n \in \mathbb{N}$ .

It was recently shown in [2] that the complex polynomial Bohnenblust–Hille inequality is, at most, subexponential, i.e., for any  $\varepsilon > 0$ , there is a constant  $C_\varepsilon > 0$  such that  $D_{\mathbb{C},m} \leq C_\varepsilon (1 + \varepsilon)^m$  for all positive integers  $m$ . The main motivation of this paper are the following problems:

- (I) Is the *real* polynomial BH inequality subexponential?
- (II) What is the optimal growth of the real polynomial BH inequality?

We provide the final answer to these previous problems by showing that

$$\limsup_m D_{\mathbb{R},m}^{1/m} = 2.$$

## 2. THE UPPER ESTIMATE

The proof of the subexponentiality of the complex BH inequality given in [2] lies heavily in arguments restricted to complex scalars (it uses, for instance estimates from [1] for complex scalars); so a simple adaptation for the real case does not work. The calculation of the upper estimate of the BH inequality is quite simplified by the use of complexifications of polynomials. In particular we are interested in the following deep result due to Visser [15], which generalizes an old result of Chebyshev:

**Theorem 2.1** (Visser, [15], 1946). *Let*

$$P(y_1, \dots, y_n) = \sum_{|\alpha| \leq m} a_\alpha y_1^{\alpha_1} \cdots y_n^{\alpha_n},$$

with  $\alpha = (\alpha_1, \dots, \alpha_n)$ ,  $|\alpha| = \alpha_1 + \cdots + \alpha_n$ , be a polynomial of total degree at most  $m \in \mathbb{N}$  in the variables  $y_1, \dots, y_n$  and with real coefficients  $a_\alpha$ . Suppose  $0 \leq k \leq m$  and  $P_k$  is the homogeneous polynomial of degree  $k$  defined by

$$P_k(y_1, \dots, y_n) = \sum_{|\alpha|=k} a_\alpha y_1^{\alpha_1} \cdots y_n^{\alpha_n}.$$

Then we have

$$\max_{z_1, \dots, z_n \in \mathbb{D}} |P_m(z_1, \dots, z_n)| \leq 2^{m-1} \cdot \max_{x_1, \dots, x_n \in [-1, 1]} |P(x_1, \dots, x_n)|,$$

where  $\mathbb{D}$  stands for the closed unit disk in  $\mathbb{C}$ . In particular, if  $P$  is homogeneous, then

$$\max_{z_1, \dots, z_n \in \mathbb{D}} |P(z_1, \dots, z_n)| \leq 2^{m-1} \cdot \max_{x_1, \dots, x_n \in [-1, 1]} |P(x_1, \dots, x_n)|.$$

Moreover, the constant  $2^{m-1}$  cannot be replaced by any smaller one.

Let  $P : \ell_\infty^n(\mathbb{R}) \rightarrow \mathbb{R}$  be an  $m$ -homogeneous polynomial

$$P(x) = \sum_{|\alpha|=m} a_\alpha x^\alpha$$

and consider the complexification  $P_{\mathbb{C}} : \ell_{\infty}^n(\mathbb{C}) \rightarrow \mathbb{C}$  of  $P$  given by

$$P_{\mathbb{C}}(z) = \sum_{|\alpha|=m} a_{\alpha} z^{\alpha}.$$

From Theorem 2.1 above we know that

$$(2.1) \quad \|P_{\mathbb{C}}\| \leq 2^{m-1} \|P\|.$$

Thus, since the complex polynomial Bohnenblust–Hille inequality is subexponential, for all  $\varepsilon > 0$  there exists  $C_{\varepsilon} > 1$  such that

$$(2.2) \quad |P|_{\frac{2m}{m+1}} = |P_{\mathbb{C}}|_{\frac{2m}{m+1}} \leq C_{\varepsilon} (1 + \varepsilon)^m \|P_{\mathbb{C}}\|$$

and combining (2.1) and (2.2) we conclude that

$$\limsup_m D_{\mathbb{R},m}^{1/m} \leq 2.$$

As we mentioned earlier, Bayart et al. proved, recently, that the complex polynomial Bohnenblust–Hille inequality is subexponential (see [2]). The following result shows that the exponential growth of the real polynomial BH inequality is sharp in a very strong way: the exponential bound can not be reduced in any sense, i.e., there is an exponential lower bound for  $D_{\mathbb{R},m}$  which holds for every  $m \in \mathbb{N}$ .

**Theorem 2.2.**

$$D_{\mathbb{R},m} > \left( \frac{2\sqrt[4]{3}}{\sqrt{5}} \right)^m > (1.17)^m$$

for all positive integers  $m > 1$ .

*Proof.* Let  $m$  be an even integer. Consider the  $m$ -homogeneous polynomial

$$R_m(x_1, \dots, x_m) = (x_1^2 - x_2^2 + x_1x_2)(x_3^2 - x_4^2 + x_3x_4) \cdots (x_{m-1}^2 - x_m^2 + x_{m-1}x_m).$$

Since  $\|R_2\| = 5/4$ , it is simple to see that

$$\|R_m\| = (5/4)^{m/2}.$$

From the BH inequality for  $R_m$  we have

$$\left( \sum_{|\alpha|=m} |a_{\alpha}|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq D_{\mathbb{R},m} \|R_m\|,$$

that is,

$$D_{\mathbb{R},m} \geq \frac{(3^{\frac{m}{2}})^{\frac{m+1}{2m}}}{(\frac{5}{4})^{\frac{m}{2}}} \geq \frac{(\sqrt{3})^{\frac{m+1}{2}}}{(\frac{5}{4})^{\frac{m}{2}}} > \left( \frac{2\sqrt[4]{3}}{\sqrt{5}} \right)^m.$$

Now let us suppose that  $m$  is odd. Keeping the previous notation, consider the  $m$  homogeneous polynomial

$$R_m(x_1, \dots, x_{2m}) = (x_{2m} + x_{2m-1}) R_{m-1}(x_1, \dots, x_{m-1}) + (x_{2m} - x_{2m-1}) R_{m-1}(x_m, \dots, x_{2m-2}).$$

So we have

$$D_{\mathbb{R},m} \geq \frac{(4 \cdot 3^{\frac{m-1}{2}})^{\frac{m+1}{2m}}}{2 \cdot (\frac{5}{4})^{\frac{m-1}{2}}} > 2^{m-1+\frac{1}{m}} \left( \frac{\sqrt[4]{3}}{\sqrt{5}} \right)^{m-1} > \left( \frac{2\sqrt[4]{3}}{\sqrt{5}} \right)^{m-1}.$$

□

## 3. THE LOWER ESTIMATE

Using our previous results, in order to show that  $\limsup_m D_{\mathbb{R},m}^{1/m} = 2$ , we just need the following theorem:

**Theorem 3.1.** *If  $k \in \mathbb{N}$  is fixed, then*

$$\limsup_m D_{\mathbb{R},m}^{1/m}(2^k) \geq 2^{1-2^{-k}}.$$

Therefore,

$$\limsup_m D_{\mathbb{R},m}^{1/m} \geq 2.$$

*Proof.* Consider the sequence of polynomials (with norm 1) defined recursively by

$$\begin{aligned} Q_2(x_1, x_2) &= x_1^2 - x_2^2, \\ Q_{2^m}(x_1, \dots, x_{2^k}) &= Q_{2^{m-1}}(x_1, \dots, x_{2^{m-1}})^2 - Q_{2^{m-1}}(x_{2^{m-1}+1}, \dots, x_{2^m})^2. \end{aligned}$$

Let us show (by induction on  $m$ ) that

$$(3.1) \quad |Q_{2^m}^n|_\infty \geq \left( \frac{2^n}{n+1} \right)^{2^m-1}$$

for every natural number  $n$ . The case  $m = 1$  comes from the fact that, since

$$2^n = \sum_{k=0}^n \binom{n}{k} \leq (n+1) \max_{0 \leq k \leq n} \binom{n}{k},$$

the  $2n$ -homogeneous polynomial  $Q_2^n$  admits the following estimate:

$$(3.2) \quad |Q_2^n|_{\frac{4n}{2n+1}} \geq |Q_2^n|_\infty = \max_{0 \leq k \leq n} \binom{n}{k} \geq \frac{2^n}{n+1}.$$

Let us now suppose that equation (3.1) holds for some  $m$ , and notice that

$$(3.3) \quad Q_{2^{m+1}}^n(x_1, x_2, \dots, x_{2^{m+1}}) = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} Q_{2^m}^{2k}(x_1, \dots, x_{2^m}) Q_{2^m}^{2(n-k)}(x_{2^m+1}, \dots, x_{2^{m+1}}).$$

The coefficient of maximal absolute value in a product of polynomials in disjoint sets of variables is the product of the respective maximal coefficients, thus

$$|Q_{2^{m+1}}^n|_\infty = \max_{0 \leq k \leq n} \binom{n}{k} |Q_{2^m}^{2k}|_\infty |Q_{2^m}^{2(n-k)}|_\infty \geq \max_{0 \leq k \leq n} \binom{n}{k} \left( \frac{2^{2n}}{(2k+1)(2n-2k+1)} \right)^{2^m-1}$$

by the induction hypothesis. However,  $(2k+1)(2n-2k+1) \leq (n+1)^2$  when  $0 \leq k \leq n$ ; thus

$$|Q_{2^{m+1}}^n|_\infty \geq \left( \frac{2^n}{n+1} \right)^{2^{m+1}-2} \max_{0 \leq k \leq n} \binom{n}{k} \geq \left( \frac{2^n}{n+1} \right)^{2^{m+1}-1},$$

by equation (3.2). Therefore, the formula given in (3.1) holds for every positive integer  $m$ . Next, every  $n$ -homogeneous polynomial  $P$  admits the clear estimate given by

$$|P|_{\frac{2n}{n+1}} \geq |P|_\infty,$$

from which equation (3.1) yields that

$$D_{\mathbb{R},n2^m}(2^m) \geq \left( \frac{2^n}{n+1} \right)^{2^m-1},$$

and the proof follows straightforwardly.  $\square$

## 4. CONTRACTIVITY IN FINITE DIMENSIONS: COMPLEX VERSUS REAL SCALARS

We remark that the complex polynomial Bohnenblust-Hille constants for polynomials on  $\mathbb{C}^n$ , with  $n \in \mathbb{N}$  fixed, are contractive.

**Proposition 4.1.** *For all  $n \geq 2$  the complex polynomial BH inequality is contractive in  $\mathcal{P}({}^m\ell_\infty^n)$ . More precisely, for all fixed  $n \in \mathbb{N}$ , there are constants  $D_m$ , with  $\lim_{m \rightarrow \infty} D_m = 1$ , so that*

$$|P|_{\frac{2m}{m+1}} \leq D_m \|P\|$$

for all  $P \in \mathcal{P}({}^m\ell_\infty^n)$ .

*Proof.* Let  $P(z) = \sum_{|\alpha|=m} c_\alpha z^\alpha$  and  $f(t) = P(e^{it_1}, \dots, e^{it_n}) = \sum_{|\alpha|=m} c_\alpha e^{i\alpha t}$ , where  $t = (t_1, \dots, t_n) \in \mathbb{R}^n$ ,  $\alpha \in (\mathbb{N} \cup \{0\})^n$  and  $\alpha t = \alpha_1 t_1 + \dots + \alpha_n t_n$ . Observe that if  $\|f\|$  denotes the sup norm of  $f$  on  $[-\pi, \pi]$ , by the Maximum Modulus Principle  $\|f\| = \|P\|$ . Also, due to the orthogonality of the system  $\{e^{iks} : k \in \mathbb{Z}\}$  in  $L^2([-\pi, \pi])$  we have

$$\|P\|^2 = \|f\|^2 \geq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^2 dt = \sum_{|\alpha|=m} |c_\alpha|^2 = |P|_2^2,$$

from which  $|P|_2 \leq \|P\|$ . On the other hand it is well known that in  $\mathbb{K}^d$  we have

$$(4.1) \quad |\cdot|_q \leq |\cdot|_p \leq d^{\frac{1}{p} - \frac{1}{q}} |\cdot|_q,$$

for all  $1 \leq p \leq q$ . Since the dimension of  $\mathcal{P}({}^m\ell_\infty^n)$  is  $\binom{m+n-1}{n-1}$ , the result follows from  $|P|_2 \leq \|P\|$  by setting in (4.1)  $p = \frac{2m}{m+1}$ ,  $q = 2$  and  $d = \binom{m+n-1}{n-1}$ . So  $D_m = \binom{m+n-1}{n-1}^{\frac{1}{2m}}$  and since

$$\lim_{m \rightarrow \infty} \binom{m+n-1}{n-1}^{\frac{1}{2m}} = 1,$$

the proof is done.  $\square$

The next result shows that the *real version* of Proposition 4.1 is not valid; we stress that Theorem 2.2 cannot be used here since it uses polynomials in a growing number of variables.

**Theorem 4.2.** *For all fixed positive integer  $N \geq 2$ , the exponentiality of the real polynomial Bohnenblust-Hille inequality in  $\mathcal{P}({}^m\ell_\infty^N)$  cannot be improved. More precisely,*

$$\limsup_m D_{\mathbb{R},m}^{1/m}(N) \geq \sqrt[8]{27} \approx 1.5098$$

for all  $N \geq 2$ .

*Proof.* It suffices to set  $N = 2$  and prove that, for  $m = 4n$ ,

$$D_{\mathbb{R},4n}(2) \geq \sqrt[4]{\frac{4}{m\pi}} \left( \sqrt[8]{27} \right)^m.$$

Consider the 4-homogeneous polynomial given by

$$P_4(x, y) = x^3 y - x y^3 = x y (x^2 - y^2).$$

A straightforward calculation shows that  $P_4$  attains its norm at  $\pm(\pm\frac{1}{\sqrt{3}}, 1)$  and  $\pm(1, \pm\frac{1}{\sqrt{3}})$  and that  $\|P_4\| = \frac{2\sqrt{3}}{9}$ . On the other hand  $\|P_4^n\| = \left(\frac{2\sqrt{3}}{9}\right)^n$  and

$$P_4(x, y)^n = x^n y^n \sum_{k=0}^n \binom{n}{k} (-1)^k x^{2k} y^{2(n-k)}.$$

Hence, if  $\mathbf{a}$  is the vector of the coefficients of  $P_4$ , using the fact that  $|\cdot|_{\frac{8n}{4n+1}} \geq |\cdot|_2$  (notice that here  $|\cdot|_2$  is the Euclidian norm), we have

$$(4.2) \quad \begin{aligned} D_{\mathbb{R},4n}(2) &\geq \frac{|\mathbf{a}|_{\frac{8n}{4n+1}}}{\|P_4\|^n} = \frac{\left[ \sum_{k=0}^n \binom{n}{k}^{\frac{8n}{4n+1}} \right]^{\frac{4n+1}{8n}}}{\left( \frac{2\sqrt{3}}{9} \right)^n} \\ &\geq \frac{\left[ \sum_{k=0}^n \binom{n}{k}^2 \right]^{\frac{1}{2}}}{\left( \frac{2\sqrt{3}}{9} \right)^n} = \frac{\sqrt{\binom{2n}{n}}}{\left( \frac{2\sqrt{3}}{9} \right)^n} = \frac{\sqrt{(2n)!}}{\left( \frac{2\sqrt{3}}{9} \right)^n n!}. \end{aligned}$$

Above we have used the well known formula

$$\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}.$$

Using Stirling's approximation formula

$$n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n$$

in (4.2) we have, for  $m = 4n$ ,

$$D_{\mathbb{R},m}(2) = D_{\mathbb{R},4n}(2) \geq \frac{\sqrt{(2n)!}}{\left( \frac{2\sqrt{3}}{9} \right)^n n!} \sim \frac{\sqrt{2\sqrt{n\pi} \left( \frac{2n}{e} \right)^{2n}}}{\left( \frac{2\sqrt{3}}{9} \right)^n \sqrt{2\pi n} \left( \frac{n}{e} \right)^n} = \sqrt[4]{\frac{4}{m\pi}} \left( \sqrt[8]{27} \right)^m.$$

□

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